# LONG TIME DECAY FOR 3D-NSE IN GEVREY-SOBOLEV SPACES

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Abstract. In this paper we prove, if $u$ is a global solution to Navier-Stokes equations in the Sobolev-Gevrey spaces $H^1_{a,\sigma}(\mathbb{R}^3)$, then $\|u(t)\|_{H^1_{a,\sigma}}$ decays to zero as time goes to infinity. Fourier analysis is used.

Contents



## 1. Introduction

The $3D$ incompressible Naviers-Stokes equations are given by:

$$(NSE) \qquad \begin{cases} \partial_t u - \Delta u + u.\nabla u &= -\nabla p \text{ in } \mathbb{R}^+ \times \mathbb{R}^3 \\ \operatorname{div} u &= 0 \text{ in } \mathbb{R}^+ \times \mathbb{R}^3 \\ u(0,x) &= u^0(x) \quad \text{in } \mathbb{R}^3, \end{cases}$$

where, we suppose that the fluid viscosity $\nu = 1$, and $u = u(t,x) = (u_1, u_2, u_3)$ and $p = p(t,x)$ denote respectively the unknown velocity and the unknown pressure of the fluid at the point $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^3$, $(u.\nabla u) := u_1\partial_1 u + u_2\partial_2 u + u_3\partial_3 u$, and $u^0 = (u_1^o(x), u_2^o(x), u_3^o(x))$ is a given initial velocity. If $u^0$ is quite regular, the divergence free condition determines the pressure $p$.

We define the Sobolev-Gevrey spaces as follows; for $a, s \geq 0$ and $\sigma > 1$,

$$H^s_{a,\sigma}(\mathbb{R}^3) = \{f \in L^2(\mathbb{R}^3);\ e^{a|D|^{1/\sigma}} f \in H^s(\mathbb{R}^3)\}.$$

It is equipped with the norm

$$\|f\|^2_{H^s_{a,\sigma}} = \|e^{a|D|^{1/\sigma}} f\|_{H^s}$$

and its associated inner product

$$\langle f/g \rangle_{H^s_{a,\sigma}} = \langle e^{a|D|^{\frac{1}{\sigma}}} f / e^{a|D|^{\frac{1}{\sigma}}} g \rangle_{H^s}.$$

There are several authors who have studied the behavior of the norm of the solution to infinity in the different Banach spaces. For example:
Wiegner proved in [9] that the $L^2$ norm of the solutions vanishes for any square integrable initial

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data, as times goes to infinity and gave a decay rate that seems to be optimal for a class of initial data. In [8, 10] M.E.Schonber and M.Wiegner derived some asymptotic properties of the solution and its higher derivatives under additional assumptions on the initial data. In [5] J.Benameur and R.Selmi proved that if $u$ be a Leray solution of $(2d-NSE)$ then $\lim_{t\to\infty}\|u(t)\|_{L^2(\mathbb{R}^2)}=0$. In [7] for the critical Sobolev spaces $\dot{H}^{\frac12}$ I.Gallagher, D.Iftimie and F.Planchon proved that $\|u(t)\|_{\dot{H}^{\frac12}}$ goes to zero at infinity. In [2] J.Benameur proved if $u\in\mathcal{C}([0,\infty),\mathcal{X}^{-1}(\mathbb{R}^3))$ be a global solution to $3D$ Navier-Stokes equation, then $\|u(t)\|_{\mathcal{X}^{-1}}$ decay to zero as times goes to infinity.

We state our main result.

**Theorem 1.1.** *Let $a>0$ and $\sigma>1$. Let $u\in\mathcal{C}([0,\infty),H^1_{a,\sigma}(\mathbb{R}^3))$ be a global solution to $(NSE)$ system. Then*

$$\limsup_{t\to\infty}\|u(t)\|_{H^1_{a,\sigma}}=0. \tag{1.1}$$

*Remark* 1.2. The existence of local solutions to $(NSE)$ was studied in a recent paper [4].

The paper is organized in the following way: In section 2, we give some notations and important preliminary results. The section 3 is devoted to prove that, if $u\in\mathcal{C}(\mathbb{R}^+,H^1(\mathbb{R}^3))$ is a global solution to $(NSE)$, then $\|u(t)\|_{H^1}$ decays to zero as time goes to infinity. This proof uses the fact that

$$\lim_{t\to\infty}\|u(t)\|_{\dot{H}^{\frac12}}=0 \tag{1.2}$$

and the energy estimate

$$\|u(t)\|_{L^2}^2+\int_0^t\|\nabla u(\tau)\|_{L^2}^2d\tau\leq\|u^0\|_{L^2}^2. \tag{1.3}$$

In section 4, we generalize the results of Foias-Temam( see [6]) to $\mathbb{R}^3$ . In section 5, we prove the main theorem. This proof is based on the obtained results in sections 3 and 4.

## 2. Notations and preliminaries results

2.1. **Notations.** In this section, we collect some notations and definitions that will be used later.
• The Fourier transformation is normalized as

$$\mathcal{F}(f)(\xi)=\widehat{f}(\xi)=\int_{\mathbb{R}^3}\exp(-ix.\xi)f(x)dx,\ \ \xi=(\xi_1,\xi_2,\xi_3)\in\mathbb{R}^3.$$

• The inverse Fourier formula is

$$\mathcal{F}^{-1}(g)(x)=(2\pi)^{-3}\int_{\mathbb{R}^3}\exp(i\xi.x)g(\xi)d\xi,\ \ x=(x_1,x_2,x_3)\in\mathbb{R}^3.$$

• For $s\in\mathbb{R}$, $H^s(\mathbb{R}^3)$ denotes the usual non-homogeneous Sobolev space on $\mathbb{R}^3$ and $\langle ./.\rangle_{H^s}$ denotes the usual scalar product on $H^s(\mathbb{R}^3)$.
• For $s\in\mathbb{R}$, $\dot{H}^s(\mathbb{R}^3)$ denotes the usual homogeneous Sobolev space on $\mathbb{R}^3$ and $\langle ./.\rangle_{\dot{H}^s}$ denotes the usual scalar product on $\dot{H}^s(\mathbb{R}^3)$.
• The convolution product of a suitable pair of functions $f$ and $g$ on $\mathbb{R}^3$ is given by

$$(f*g)(x):=\int_{\mathbb{R}^3}f(y)g(x-y)dy.$$

• If $f=(f_1,f_2,f_3)$ and $g=(g_1,g_2,g_3)$ are two vector fields, we set

$$f\otimes g:=(g_1f,g_2f,g_3f),$$

and

$$\mathrm{div}\,(f\otimes g):=(\mathrm{div}\,(g_1f),\mathrm{div}\,(g_2f),\mathrm{div}\,(g_3f)).$$

## 2.2. **Preliminary results.**

**Lemma 2.1.** *(See [1]) Let $(s,t)\in\mathbb{R}^2$, such that $s<\frac{3}{2}$ and $s+t>0$. Then, there exists a constant $C>0$, such that*

$$\|uv\|_{\dot H^{s+t-\frac{3}{2}}(\mathbb{R}^3)}\leq C(\|u\|_{\dot H^{s}(\mathbb{R}^3)}\|v\|_{\dot H^{t}(\mathbb{R}^3)}+\|u\|_{\dot H^{t}(\mathbb{R}^3)}\|v\|_{\dot H^{s}(\mathbb{R}^3)}).$$

*If $s<\frac{3}{2}$, $t<\frac{3}{2}$ and $s+t>0$, then there exists a constant $C>0$, such that*

$$\|uv\|_{\dot H^{s+t-\frac{3}{2}}(\mathbb{R}^3)}\leq C\|u\|_{\dot H^{s}(\mathbb{R}^3)}\|v\|_{\dot H^{t}(\mathbb{R}^3)}.$$

**Lemma 2.2.** *Let $f\in\dot H^{s_1}(\mathbb{R}^3)\cap\dot H^{s_2}(\mathbb{R}^3)$, where $s_1<\frac{3}{2}<s_2$. Then, there is a constant $c=c(s_1,s_2)$ such that*

$$\|f\|_{L^\infty(\mathbb{R}^3)}\leq\|\widehat{f}\|_{L^1(\mathbb{R}^3)}\leq c\|f\|_{\dot H^{s_1}(\mathbb{R}^3)}^{\frac{s_2-\frac{3}{2}}{s_2-s_1}}\|f\|_{\dot H^{s_2}(\mathbb{R}^3)}^{\frac{\frac{3}{2}-s_1}{s_2-s_1}}.$$

*Proof.* We have

$$\begin{aligned}\|f\|_{L^\infty}&\leq\|\widehat{f}\|_{L^1}\\&\leq\int_\xi|\widehat{f(\xi)}|d\xi\\&\leq\int_{|\xi|<\lambda}|\widehat{f(\xi)}|d\xi+\int_{|\xi|>\lambda}|\widehat{f(\xi)}|d\xi.\end{aligned}$$

We take

$$I_1=\int_{|\xi|<\lambda}\frac{1}{|\xi|^{s_1}}|\xi|^{s_1}|\widehat{f(\xi)}|d\xi.$$

Using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}I_1&\leq\left(\int_{|\xi|<\lambda}\frac{1}{|\xi|^{s_1}}d\xi\right)^{\frac{1}{2}}\|f\|_{\dot H^{s_1}(\mathbb{R}^3)}\\&\leq\left(\int_0^\lambda\frac{1}{r^{2s_1-2}}dr\right)^{\frac{1}{2}}\|f\|_{\dot H^{s_1}(\mathbb{R}^3)}\\&\leq c_{s_1}\lambda^{\frac{3}{2}-s_1}\|f\|_{\dot H^{s_1}(\mathbb{R}^3)}.\end{aligned}$$

Similarly, take

$$I_2=\int_{|\xi|>\lambda}\frac{1}{|\xi|^{s_2}}|\xi|^{s_2}|\widehat{f(\xi)}|d\xi,$$

we have

$$\begin{aligned}I_2&\leq\left(\int_{|\xi|>\lambda}\frac{1}{|\xi|^{s_2}}d\xi\right)^{\frac{1}{2}}\|f\|_{\dot H^{s_2}}\\&\leq\left(\int_\lambda^\infty\frac{1}{r^{2s_2-2}}dr\right)^{\frac{1}{2}}\|f\|_{\dot H^{s_2}}\\&\leq c_{s_2}\lambda^{\frac{3}{2}-s_2}\|f\|_{\dot H^{s_2}}.\end{aligned}$$

Therefore,

$$\|f\|_{L^\infty}\leq A\lambda^{\frac{3}{2}-s_1}+B\lambda^{\frac{3}{2}-s_2}.$$

with $A=c_{s_1}\|f\|_{\dot H^{s_1}}$ and $B=c_{s_2}\|f\|_{\dot H^{s_2}}$.
Posing

$$\varphi(\lambda)=A\lambda^{\frac{3}{2}-s_1}+B\lambda^{\frac{3}{2}-s_2}.$$

Then, $\varphi'(\lambda) = 0 \Leftrightarrow \lambda = c(s_1, s_2)\left(\frac{B}{A}\right)^{\frac{1}{s_2-s_1}}$
So,

$$||f||_{L^\infty(\mathbb{R}^3)} \leq c' A^{\frac{s_2-\frac{3}{2}}{s_2-s_1}} B^{\frac{\frac{3}{2}-s_1}{s_2-s_1}}.$$

$\square$

*Remark* 2.3. In particular, for $s_1 = 1$ and $s_2 = 2$, where $f \in \dot{H}^1(\mathbb{R}^3) \cap \dot{H}^2(\mathbb{R}^3)$, we get

$$||f||_{L^\infty} \leq ||f||_{\dot{H}^1}^{\frac{1}{2}} ||f||_{\dot{H}^2}^{\frac{1}{2}}.$$

## 3. Long time decay of $(NSE)$ system in $H^1(\mathbb{R}^3)$

In this section, we want to prove: If $u \in \mathcal{C}(\mathbb{R}^+, H^1(\mathbb{R}^3))$ is a global solution to $(NSE)$ system, then

$$\limsup_{t\to\infty} \|u(t)\|_{H^1} = 0. \tag{3.1}$$

This proof is done in two steps.

• Step 1: In this step, we shall prove that

$$\limsup_{t\to\infty} \|u(t)\|_{\dot{H}^1} = 0. \tag{3.2}$$

We have

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{\dot{H}^{\frac{1}{2}}} + \|u\|^2_{\dot{H}^{\frac{3}{2}}} \leq c\|u\|_{\dot{H}^{\frac{1}{2}}}\|u\|^2_{\dot{H}^{\frac{3}{2}}}.$$

From (1.2), let $t_0 > 0$ such that $\|u(t_0)\|_{\dot{H}^{\frac{1}{2}}} < \frac{1}{2c}$. Then

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{\dot{H}^{\frac{1}{2}}} + \frac{1}{2}\|u\|^2_{\dot{H}^{\frac{3}{2}}} \leq 0, \;\; \forall t \geq t_0.$$

Integrating with respect to time, we obtain

$$\|u(t)\|^2_{\dot{H}^{\frac{1}{2}}} + \int_{t_0}^t \|u(\tau)\|^2_{\dot{H}^{\frac{3}{2}}} \leq \|u(t_0)\|^2_{\dot{H}^{\frac{1}{2}}}, \;\; \forall t \geq t_0.$$

Let $s > 0$ and $c = c_s$. There exists $T_0 = T_0(s, \nu, u^0) > 0$, such that

$$\|u(T_0)\|_{\dot{H}^{\frac{1}{2}}} < \frac{1}{2c_s}.$$

Then

$$\|u(t)\|_{\dot{H}^{\frac{1}{2}}} < c_s, \;\; \forall t \geq t_0.$$

Now, for $s > 0$ we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u\|^2_{\dot{H}^s} + \|u\|^2_{\dot{H}^{s+1}} &\leq \|u\otimes u\|_{\dot{H}^s}\|u\|_{\dot{H}^{s+1}} \\ &\leq c_s\|u\|_{\dot{H}^{\frac{1}{2}}}\|u\|^2_{\dot{H}^{s+1}}.\end{aligned}$$

Then

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{\dot{H}^s} + \|u\|^2_{\dot{H}^{s+1}} \leq \frac{\nu}{2}\|u\|^2_{\dot{H}^{s+1}}, \;\; \forall t \geq T_0.$$

Thus

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{\dot{H}^s} + \frac{1}{2}\|u(t)\|^2_{\dot{H}^{s+1}} \leq 0, \;\; \forall t \geq T_0.$$

So, for $T_0 \leq t' \leq t$,

$$\|u(t)\|^2_{\dot{H}^s} + \int_{t'}^t \|u(\tau)\|^2_{\dot{H}^{s+1}} d\tau \leq \|u(t')\|^2_{\dot{H}^s}.$$

In particular, for $s = 1$

$$\|u(t)\|^2_{\dot{H}^1} + \int_{t'}^t \|u(\tau)\|^2_{\dot{H}^2} d\tau \leq \|u(t')\|^2_{\dot{H}^1}.$$

Then $(t \to \|u(t)\|_{\dot{H}^1})$ is decreasing on $[T_0, \infty)$ and $u \in L^2([0,\infty), \dot{H}^2(\mathbb{R}^3))$.
Now, let $\varepsilon > 0$ small enough. The $L^2$-energy estimate

$$\|u(t)\|_{L^2}^2 + 2\int_{T_0}^t \|\nabla u(\tau)\|_{L^2}^2 d\tau \leq \|u(T_0)\|_{L^2}^2, \ \ \forall t \geq T_0$$

implies that $u \in L^2([T_0,\infty), \dot{H}^1(\mathbb{R}^3))$ and there is a time $t_\varepsilon \geq T_0$ such that

$$\|u(t_\varepsilon)\|_{\dot{H}^1} < \varepsilon.$$

As $(t \longrightarrow \|u(t)\|_{\dot{H}^1})$ is decreasing on $[T_0, \infty)$, then

$$\|u(t)\|_{\dot{H}^1} < \varepsilon, \ \forall t \geq t_\varepsilon.$$

Therefore (3.2) is proved. □

• Step 2: In this step, we prove that

$$\limsup_{t\to\infty} \|u(t)\|_{L^2} = 0. \tag{3.3}$$

This proof is inspired by [3] and [5]. For $\delta > 0$ and a given distribution $f$, we define the operators $A_\delta(D)$ and $B_\delta(D)$, as following:

$$A_\delta(D)f = \mathcal{F}^{-1}(\mathbf{1}_{\{|\xi|<\delta\}}\mathcal{F}(f)), \ B_\delta(D)f = \mathcal{F}^{-1}(\mathbf{1}_{\{|\xi|\geq\delta\}}\mathcal{F}(f)).$$

It is clear that when applying $A_\delta(D)$ (respectively, $B_\delta(D)$) to any distribution, we are dealing with its low-frequency part(respectively, high- frequency part).
Let $u$ be a solution to $(NSE)$. Denote by $\omega_\delta$ and $v_\delta$, respectively, the low-frequency part and the high-frequency part of $u$ and so on ${\omega_\delta}^0$ and ${v_\delta}^0$ for the initial data $u^0$. Applying the pseudo-differential operators $A_\delta(D)$ to the $(NSE)$, we get

$$\partial_t\omega_\delta - \nu\triangle\omega_\delta + A_\delta(D)\mathbb{P}(u.\nabla u) = 0.$$

Taking the $L^2(\mathbb{R}^3)$ inner product, we obtain

$$\begin{array}{rcl}
\frac{1}{2}\frac{d}{dt}\|\omega_\delta(t)\|_{L^2}^2 + \|\nabla\omega_\delta(t)\|_{L^2}^2 & \leq & |\langle A_\delta(D)\mathbb{P}(u.\nabla u)/\omega_\delta(t)\rangle_{L^2}| \\
& \leq & |\langle A_\delta(D)(\mathrm{div}\,(u\otimes u)/\omega_\delta(t)\rangle_{L^2}| \\
& \leq & |\langle A_\delta(D)(u\otimes u))/\nabla\omega_\delta(t)\rangle_{L^2}| \\
& \leq & |\langle u\otimes u/\nabla\omega_\delta(t)\rangle_{L^2}| \\
& \leq & \|u\otimes u\|_{L^2}\|\nabla\omega_\delta(t)\|_{L^2} \\
& \leq & \|u\otimes u\|_{L^2}\|\nabla\omega_\delta(t)\|_{L^2}.
\end{array}$$

Lemma 2.1 yields

$$\begin{array}{rcl}
\frac{1}{2}\frac{d}{dt}\|\omega_\delta(t)\|_{L^2}^2 + \|\nabla\omega_\delta(t)\|_{L^2}^2 & \leq & C\|u(t)\|_{\dot{H}^{\frac{1}{2}}}\|\nabla u(t)\|_{L^2}\|\nabla\omega_\delta(t)\|_{L^2} \\
& \leq & CM\|\nabla u(t)\|_{L^2}\|\nabla\omega_\delta(t)\|_{L^2} \quad (M = \sup_{t\geq 0}\|u(t)\|_{\dot{H}^{\frac{1}{2}}}).
\end{array}$$

Integrating with respect to time, we obtain

$$\|\omega_\delta(t)\|_{L^2}^2 \leq \|{\omega_\delta}^0\|_{L^2}^2 + CM\int_0^t \|\nabla u(\tau)\|_{L^2}\|\nabla\omega_\delta(\tau)\|_{L^2}d\tau.$$

Hence, we have $\|\omega_\delta(t)\|_{L^2}^2 \leq M_\delta$ for all $t \geq 0$, where

$$M_\delta = \|{\omega_\delta}^0\|_{L^2}^2 + CM\int_0^\infty \|\nabla u(\tau)\|_{L^2}\|\nabla\omega_\delta(\tau)\|_{L^2}d\tau.$$

On the one hand, it is clear that $\lim_{\delta\to 0}\|{\omega_\delta}^0\|_{L^2(\mathbb{R}^3)}^2 = 0$. On the other hand, the Lebesgue-Dominated Convergence Theorem implies that

$$\lim_{\delta\to 0}\int_0^\infty \|\nabla u(\tau)\|_{L^2}\|\nabla\omega_\delta(\tau)\|_{L^2}d\tau = 0. \tag{3.4}$$

Hence $\lim_{\delta\to 0} M_\delta = 0$, and thus

$$\lim_{\delta\to 0}\sup_{t\geq 0}\|\omega_\delta(t)\|_{L^2} = 0. \tag{3.5}$$

At this point, we note that it makes sense to take time equal to $\infty$ in the integral (3.4). In fact, by definition of $\omega_\delta$ we have $\|\nabla\omega_\delta\|_{L^2} \leq \|\nabla u\|_{L^2}$ .
It is clear that, $\lim_{\delta\to 0}\|\nabla\omega_\delta(t)\|_{L^2} = 0$ almost everywhere. So, the integrand sequence

$$\|\nabla u(t)\|_{L^2}\|\nabla\omega_\delta(t)\|_{L^2}$$

converges point-wise to zero. Moreover, using the above computations and (1.3), we obtain

$$\|\nabla u(t)\|_{L^2}\|\nabla\omega_\delta(t)\|_{L^2} \leq \|\nabla u(t)\|^2_{L^2} \in L^1(\mathbb{R}^+).$$

Thus, the integral sequence is dominated by an integrable function. Then the limiting function is integrable and one can take the time $T = \infty$ in (3.4).
Now, let us investigate the high-frequency part. To do so, one applies the pseudo-differential operators $B_\delta(D)$ to the $(NSE)$ to get

$$\partial_t v_\delta - \Delta v_\delta + B_\delta(D)\mathbb{P}(u.\nabla u) = 0.$$

Taking the Fourier transform with respect to the space variable, we obtain

$$\begin{aligned}
\partial_t|\widehat{v_\delta}(t,\xi)|^2 + 2|\xi|^2|\widehat{v_\delta}(t,\xi)|^2 &\leq 2|\mathcal{F}(B_\delta(D)\mathbb{P}(u.\nabla u))(t,\xi)||\widehat{v_\delta}(t,\xi)|\\
&\leq 2|\xi||\mathcal{F}(B_\delta(D)\mathbb{P}(u\otimes u))(t,\xi)|.|\widehat{v_\delta}(t,\xi)|\\
&\leq 2|\mathcal{F}(u\otimes u)(t,\xi)|.|\widehat{\nabla v_\delta}(t,\xi)|.
\end{aligned}$$

Multiplying the obtained equation by $\exp(2\nu|\xi|^2)$ and integrating with respect to time, we get

$$|\widehat{v_\delta}(t,\xi)|^2 \leq e^{-2t|\xi|^2}|\widehat{v^0_\delta}(\xi)|^2 + 2\int_0^t e^{-2(t-\tau)|\xi|^2}|\mathcal{F}(u\otimes u)(\tau,\xi)|.|\widehat{\nabla v_\delta}(\tau,\xi)|d\tau.$$

Since $|\xi| > \delta$, we have

$$|\widehat{v_\delta}(t,\xi)|^2 \leq e^{-2t\delta^2}|\widehat{v^0_\delta}(\xi)|^2 + 2\int_0^t e^{-2(t-\tau)\delta^2}|\mathcal{F}(u\otimes u)(\tau,\xi)|.|\widehat{\nabla v_\delta}(\tau,\xi)|d\tau.$$

Integrating with respect to the frequency variable $\xi$ and using Cauchy-Schwartz inequality, we obtain

$$\|v_\delta(t)\|^2_{L^2} \leq e^{-2t\delta^2}\|v_{\delta^0}\|^2_{L^2} + 2\int_0^t e^{-2(t-\tau)\delta^2}\|u\otimes u\|_{L^2}\|\nabla v_\delta\|_{L^2}d\tau.$$

By the definition of $v_\delta$, we have

$$\|v_\delta(t)\|^2_{L^2} \leq e^{-2t\delta^2}\|u^0\|^2_{L^2} + 2\int_0^t e^{-2(t-\tau)\delta^2}\|u\otimes u\|_{L^2}\|\nabla u\|_{L^2}d\tau.$$

Lemma 2.1 and inequality (1.2) yield

$$\begin{aligned}
\|v_\delta(t)\|^2_{L^2(\mathbb{R}^3)} &\leq e^{-2t\delta^2}\|u^0\|^2_{L^2(\mathbb{R}^3)} + c\int_0^t e^{-2(t-\tau)\delta^2}\|u\|_{\dot H^{\frac12}}\|\nabla u\|^2_{L^2}d\tau.\\
&\leq e^{-2t\delta^2}\|u^0\|^2_{L^2} + CM\int_0^t e^{-2(t-\tau)\delta^2}\|\nabla u\|^2_{L^2}d\tau,\ \ (M = \sup_{t\geq 0}\|u\|_{\dot H^{\frac12}}).
\end{aligned}$$

Hence, $\|v_\delta(t)\|^2_{L^2} \leq N_\delta(t)$, where

$$N_\delta(t) = e^{-2t\delta^2}\|u^0\|^2_{L^2} + CM\int_0^\infty e^{-2(t-\tau)\delta^2}\|\nabla u\|^2_{L^2}d\tau.$$

Using Young inequality and inequality (1.3), we get $N_\delta \in L^1(\mathbb{R}^+)$ and

$$\int_0^\infty N_\delta(t)dt \leq \frac{\|u^0\|^2_{L^2}}{2\delta^2} + \frac{CM\|u^0\|^2_{L^2}}{4\delta^2}.$$

So $t \to \|\upsilon_\delta(t)\|_{L^2}^2$ is continuous and belongs to $L^1(\mathbb{R}^+)$.
Now, let $\varepsilon > 0$. At first, ( 3.5) implies that there exist some $\delta_0 > 0$ such that

$$\|\omega_{\delta_0}(t)\|_{L^2} \leq \varepsilon/2, \, \forall\, t \geq 0.$$

Let us consider the set $\mathrm{R}_{\delta_0}$ defined by $\mathrm{R}_{\delta_0} := \{t \geq 0, \, \|\upsilon_\delta(t)\|_{L^2(\mathbb{R}^3)} > \varepsilon/2\}$. If we denote by $\lambda_1(\mathrm{R}_{\delta_0})$ the Lebesgue measure of $\mathrm{R}_{\delta_0}$, we have

$$\int_0^\infty \|\upsilon_{\delta_0}(t)\|_{L^2(\mathbb{R}^3)}^2 dt \geq \int_{\mathrm{R}_{\delta_0}} \|\upsilon_\delta(t)\|_{L^2(\mathbb{R}^3)}^2 dt \geq (\varepsilon/2)^2 \lambda_1(\mathrm{R}_{\delta_0}).$$

By doing this, we can deduce that $\lambda_1(\mathrm{R}_{\delta_0}) = T^\varepsilon_{\delta^0} < \infty$, and there exists $t^\varepsilon_{\delta^0} > T^\varepsilon_{\delta^0}$ such that

$$\|\upsilon_{\delta_0}(t^\varepsilon_{\delta^0})\|_{L^2}^2 \leq (\varepsilon/2)^2.$$

So, $\|u(t^\varepsilon_{\delta^0})\|_{L^2} \leq \varepsilon$ and from (1.3) we have

$$\|u(t)\|_{L^2} \leq \varepsilon, \;\; \forall t \geq t^\varepsilon_{\delta^0}.$$

This completes the proof of (3.3). □

## 4. Generalization of Foias-Temam result in $H^1(\mathbb{R}^3)$

In [6] Fioas and Teamam proved an analytic property for the Navier-Stokes equations on the torus $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$. Here, we give a similar result on whole space $\mathbb{R}^3$.

**Theorem 4.1.** *We assume that $u^0 \in H^1(\mathbb{R}^3)$. Then, there exists a time $T$ that depends only on the $\|u^0\|_{\dot{H}^1(\mathbb{R}^3)}$, such that:*
*$(NSE)$ possesses on $(0,T)$ a unique regular solution $u$ such that $(t \to e^{\nu t|D|}u(t))$ is continuous from $[0,T]$ into $H^1(\mathbb{R}^3)$). Moreover if $u \in \mathcal{C}(\mathbb{R}^+, H^1(\mathbb{R}^3))$ is a global solution to $(NSE)$ system, then there are $M \geq 0$ and $t_0 > 0$ such that*

$$\|e^{t_0|D|}u(t)\|_{H^1(\mathbb{R}^3)} \leq M, \;\; \forall t \geq t_0.$$

Before proving this theorem, we need the following lemmas

**Lemma 4.2.** *Let $t \mapsto e^{t|D|}u \in H^2(\mathbb{R}^3)$, where $|D| = (\Delta)^{\frac{1}{2}}$. Then*

$$\|e^{t|D|}u.\nabla v\|_{L^2(\mathbb{R}^3)} \leq \|e^{t|D|}u\|_{H^1(\mathbb{R}^3)}^{\frac{1}{2}} \|e^{t|D|}u\|_{H^2(\mathbb{R}^3)}^{\frac{1}{2}} \|e^{t|D|}\Delta^{\frac{1}{2}}v\|_{L^2(\mathbb{R}^3)}.$$

*Proof.* We have

$$\begin{array}{rcl}
\|e^{t|D|}u.\nabla v\|_{L^2} & = & \displaystyle\int_{\mathbb{R}^3} e^{2t|\xi|}|\widehat{u.\nabla v}(\xi)|^2 d\xi \\
& \leq & \displaystyle\int_{\mathbb{R}^3} e^{2t|\xi|}\left(\int_{\mathbb{R}^3} |\widehat{u}(\xi-\eta)||\widehat{\nabla}v(\eta|d\eta\right)^2 d\xi \\
& \leq & \displaystyle\int_{\mathbb{R}^3}\left(\int_{\mathbb{R}^3} e^{t|\xi|}|\widehat{u}(\xi-\eta)||\widehat{\nabla}v(\eta|d\eta\right)^2 d\xi \\
& \leq & \displaystyle\int_{\mathbb{R}^3}\left(\int_{\mathbb{R}^3}\left(e^{t|\xi-\eta|}|\widehat{u}(\xi-\eta)|\right)\left(e^{t|\eta|}|\eta||\widehat{v}(\eta)|\right)d\eta\right)^2 d\xi \\
& \leq & \displaystyle\left(\int_{\mathbb{R}^3} e^{t|\xi|}|\widehat{u}(\xi)|d\xi\right)^2 ||e^{t|D|}\Delta^{\frac{1}{2}}v||_{L^2}.
\end{array}$$

Hence, for $f = \mathcal{F}^{-1}(e^{t|\xi|}|\widehat{u}(\xi)|) \in H^2(\mathbb{R}^3)$ and $(s_1 = 1; \; s_2 = 2)$, lemma 2.2 gives the desired result.

**Lemma 4.3.** *Let $t \mapsto e^{t|D|}u \in H^2(\mathbb{R}^3)$. Then*

$$\left|\langle e^{t|D|}(u.\nabla v)/e^{t|D|}w\rangle_{H^1}\right| \leq \|e^{t|D|}u\|_{H^1}^{\frac{1}{2}}\|e^{t|D|}u\|_{H^2}^{\frac{1}{2}}\|e^{t|D|}\Delta^{\frac{1}{2}}v\|_{L^2}\|e^{t|D|}\Delta w\|_{L^2}.$$

*Proof.*
We have

$$\begin{aligned}\langle u.\nabla v/w\rangle_{H^1} &= \sum_{|j|=1}\langle \partial_j(u.\nabla v)/\partial_j w\rangle_{L^2}\\ &= -\sum_{|j|=1}\langle u.\nabla v/\partial_j^2 w\rangle_{L^2}\\ &= -\sum_{|j|=1}\langle u.\nabla v/\Delta w\rangle_{L^2}.\end{aligned}$$

Then

$$\begin{aligned}\Big|\langle e^{t|D|}u.\nabla v/e^{t|D|}w\rangle_{H^1}\Big| &= \Big|\langle e^{t|D|}u.\nabla v/e^{t|D|}\Delta w\rangle_{L^2}\Big|\\ &\leq \|e^{t|D|}u.\nabla v\|_{L^2}\|e^{t|D|}\Delta w\|_{L^2}\end{aligned}$$

Finally, using lemma 4.2, we obtain the desired result.

*Proof of theorem 4.1.* We have

$$\partial_t u - \Delta u + u.\nabla u = -\nabla p. \tag{4.1}$$

Applying the fourier transform to the last equation and multiplying by $\overline{\widehat{u}}$, we have

$$\partial_t\widehat{u}.\overline{\widehat{u}} + |\xi|^2|\widehat{u}|^2 = -(\widehat{u.\nabla u}).\overline{\widehat{u}}. \tag{4.2}$$

Again, the fourier (bar) of (4.1) multiplied by $\widehat{u}$ gives

$$\partial_t\overline{\widehat{u}}.\widehat{u} + |\xi|^2|\widehat{u}|^2 = -(\overline{\widehat{u.\nabla u}}).\widehat{u}. \tag{4.3}$$

Hence, the sum of (4.2) and (4.3) yields

$$\partial_t|\widehat{u}|^2 + 2|\xi|^2|\widehat{u}|^2 = -2Re((\widehat{u.\nabla u}).\widehat{u}).$$

This implies

$$\partial_t|\widehat{u}|^2(1+|\xi|^2)e^{2t|\xi|} + 2(1+|\xi|^2)|\xi|^2e^{2t|\xi|}|\widehat{u}|^2 = -2Re((\widehat{u.\nabla u}).\widehat{u})(1+|\xi|^2)e^{2t|\xi|}.$$

Then

$$\int_{\mathbb{R}^3}(1+|\xi|^2)e^{2t|\xi|}\partial_t|\widehat{u}(\xi)|^2d\xi + 2\int_{\mathbb{R}^3}(1+|\xi|^2)|\xi|^2e^{2t|\xi|}|\widehat{u}(\xi)|^2d\xi = -2Re\int_{\mathbb{R}^3}((\widehat{u.\nabla u}).\widehat{u})(1+|\xi|^2)e^{2t|\xi|}d\xi.$$

Thus

$$\langle e^{t|D|}\partial_t u/e^{t|D|}u\rangle_{H^1} + 2\|e^{t|D|}\nabla u\|^2_{H^1(\mathbb{R}^3)} = -2Re\langle e^{t|D|}(u.\nabla u)/e^{t|D|}u\rangle_{H^1}.$$

At time $\tau$, we have

$$\langle e^{\tau|D|}u'(\tau)/e^{\tau|D|}u(\tau)\rangle_{H^1} + 2\|e^{\tau|D|}\nabla u\|^2_{H^1} = -2Re\langle e^{\tau|D|}(u.\nabla u)/e^{t|D|}u\rangle_{H^1}. \tag{4.4}$$

Therefore

$$\begin{aligned}\langle e^{t|D|}u'(t)/e^{t|D|}u(t)\rangle_{H^1} &= \langle (e^{t|D|}u(t))' - |D|e^{t|D|}u(t)/e^{t|D|}u(t)\rangle_{H^1}\\ &= \frac{1}{2}\frac{d}{dt}\|e^{t|D|}u\|^2_{H^1} - \langle e^{t|D|}|D|u(t)/e^{t|D|}u(t)\rangle_{H^1}\\ &\geq \frac{1}{2}\frac{d}{dt}\|e^{t|D|}u\|^2_{H^1} - \|e^{t|D|}u\|_{H^1}\|e^{t|D|}u\|_{H^2}.\end{aligned}$$

Using the Young inequality, we obtain

$$\frac{d}{dt}\|e^{t|D|}u\|^2_{H^1} - 2\|e^{t|D|}u\|^2_{H^1} - \frac{1}{2}\|e^{t|D|}u\|^2_{H^2} \leq 2\langle e^{t|D|}u'(t)/e^{t|D|}u(t)\rangle_{H^1}. \tag{4.5}$$

Hence, using the lemma 4.3 and Young inequality the right hand of (4.4) satisfies

$$\begin{aligned}|-2Re\langle e^{\tau|D|}u\nabla u/e^{\tau|D|}u\rangle_{H^1}| &\leq 2\|e^{\tau|D|}u\|_{H^1}^{\frac{1}{2}}\|e^{\tau|D|}u\|_{H^1}^{\frac{1}{2}}\|e^{\tau|D|}|D|u\|_{L^2}\|e^{\tau|D|}\Delta u\|_{L^2}\\ &\leq 2\|e^{\tau|D|}u\|_{H^1}^{\frac{3}{2}}\|e^{\tau|D|}u\|_{H^2}^{\frac{3}{2}}\\ &\leq \frac{3}{4}\|e^{\tau|D|}u\|_{H^2}^2+\frac{c_1}{2}\|e^{\tau|D|}u\|_{H^1}^6,\end{aligned}$$

where $c_1$ is a positive constant.
Then (4.4) yields

$$\langle e^{t|D|}u'(t)/e^{t|D|}u(t)\rangle_{H^1}+2\|e^{t|D|}\nabla u\|_{H^1}^2 \leq \frac{3}{4}\|e^{t|D|}u\|_{H^2}^2+\frac{c_1}{2}\|e^{t|D|}u\|_{H^1}^6. \tag{4.6}$$

Hence, using (4.5) and (5.1), we get

$$\begin{aligned}\frac{d}{dt}\|e^{t|D|}u\|_{H^1}^2+2\|e^{t|D|}\nabla u\|_{H^1}^2 &\leq 4\|e^{t|D|}u\|_{H^1}^2+c_1\|e^{t|D|}u\|_{H^1}^6\\ &\leq c_2+2c_1\|e^{t|D|}u\|_{H^1}^6,\end{aligned}$$

where also $c_2$ is a positive constant.
Finally, we obtain

$$y'(t)\leq K_1y^3(t),$$

where

$$y(t)=1+\|e^{t|D|}u(t)\|_{H^1}^2 \quad \text{and} \quad K_1=2c_1+c_2.$$

Then

$$y(t)\leq y(0)+K_1\int_0^t y^3(s)ds.$$

Let

$$T_1=\frac{2}{K_1y^2(0)}$$

and $o<T\leq T^*$ such that $T=\sup\{t\in[0,T^*)\,|\,\sup_{0\leq s\leq t}y(s)\leq 2y(0)\}$. Hence for $0\leq t\leq \min(T_1,T)$, we have

$$\begin{aligned}y(t) &\leq y(0)+K_1\int_0^t y^3(s)ds\\ &\leq y(0)+K_1\int_0^t 8y^3(0)ds\\ &\leq \left(1+K_18T_1y^2(0)\right)y(0).\end{aligned}$$

Taking $1+K_18T_1y^2(0)<2$, we get $T>T_1$. Then

$$y(t)\leq 2y(0),\ \forall t\in[0,T_1].$$

Therefore $t\mapsto e^{t|D|}u(t)\in H^{1(\mathbb{R}^3)},\ \forall t\in[0,T_1]$.
In particular

$$\|e^{T_1|D|}u(T_1)\|_{H^1}^2\leq 2+2\|u_0\|_{H^1}^2.$$

Now, if we know that

$$\|u(t)\|_{H^1}\leq M_1\ \ \forall t\geq 0.$$

Defining the system

$$\begin{cases}\partial_tw-\Delta w+w.\nabla w &= -\nabla p_2 \text{ in } \mathbb{R}^+\times\mathbb{R}^3,\\ \operatorname{div} w &= 0 \text{ in } \mathbb{R}^+\times\mathbb{R}^3,\\ w(0) &= u(b)\quad \text{in } \mathbb{R}^3,\end{cases}$$

where $w(t) = u(T+t)$.
Using a similar technic, we can prove that there exists $T_2 = \frac{2}{K_1}(1+M_1^2)^{-2}$ such that

$$y(t) = 1 + \|e^{t|D|}w(t)\|_{H^1}^2 \le 2(1+M_1^2), \ \ \forall t \in [0, T_2].$$

This implies that $1 + \|e^{t|D|}u(T+t)\|_{H^1}^2 \le 2(1+M_1^2)$. Hence, for $t = T_2$ we have

$$\|e^{T_2|D|}u(T+T_2)\|_{H^1}^2 \le 2(1+M_1^2).$$

Since $t = T + T_2 \ge T_2, \ \ \forall T \ge 0$, we obtain

$$\|e^{T_2|D|}u(t)\|_{H^1}^2 \le 2(1+M_1^2), \ \ \forall t \ge T_2.$$

Then

$$\|e^{T_2|D|}u(t)\|_{H^1}^2 \le 2(1+M_1^2), \ \ \forall t \ge T_2,$$

where

$$T_2 = T_2(M_1) = \frac{2}{K_1}(1+M_1^2)^{-2}.$$

□

## 5. Proof of main result

In this section, we prove the main theorem 1.1. This proof uses the result of sections 3 and 4. Let $u \in \mathcal{C}(\mathbb{R}^+, H^1_{a,\sigma}(\mathbb{R}^3))$. As $H^1_{a,\sigma}(\mathbb{R}^3) \hookrightarrow H^1(\mathbb{R}^3)$, then $u \in \mathcal{C}(\mathbb{R}^+, H^1(\mathbb{R}^3))$.
Applying the theorem 4.1, there exist $t_0 >$ and $\alpha > 0$ such that

$$\|e^{\alpha|D|}u(t)\|_{H^1} \le c_0 = 2 + M_1^2, \quad \forall t \ge t_0, \tag{5.1}$$

where $\alpha = \varphi(t_0)$ and $t_0 = \frac{2}{K_1}(1+M_1^2)^{-2}$.
Therefore, let $a > 0$, $\beta > 0$. It shows that there exists $c_3 \ge 0$ such that

$$ax^{\frac{1}{\sigma}} \le c_3 + \beta x, \ \ \forall x \ge 0.$$

Indeed; $\frac{1}{\sigma} + \frac{\sigma-1}{\sigma} = \frac{1}{p} + \frac{1}{q} = 1$. Using the Young inequality, we obtain

$$\begin{aligned} ax^{\frac{1}{\sigma}} &= a\beta^{\frac{-1}{\sigma}}(\beta^{\frac{1}{\sigma}}x^{\frac{1}{\sigma}}) \\ &\le \frac{(a\beta^{\frac{-1}{\sigma}})^q}{q} + \frac{(\beta^{\frac{1}{\sigma}}x^{\frac{1}{\sigma}})^p}{p} \\ &\le c_3 + \frac{\beta x}{\sigma} \\ &\le c_3 + \beta x, \end{aligned}$$

where $c_3 = \frac{\sigma-1}{\sigma}a^{\frac{\sigma}{\sigma-1}}\beta^{\frac{1}{1-\sigma}}$.
Take $\beta = \frac{\alpha}{2}$, using (5.1) and the Cauchy Schwarz inequality, we have

$$\begin{aligned} \|u(t)\|_{H^1_{a,\sigma}} &= \|e^{a|D|^{1/\sigma}}u(t)\|_{H^1} \\ &= \int (1+|\xi|^2)e^{2a|\xi|^{1/\sigma}}|\widehat{u}(t,\xi)|^2 d\xi \\ &= \int (1+|\xi|^2)e^{2(c_3+\beta|\xi|})|\widehat{u}(t,\xi)|^2 d\xi \\ &= \int (1+|\xi|^2)e^{2c_3}e^{\alpha|\xi|}|\widehat{u}(t,\xi)|^2 d\xi \\ &\le e^{2c_3}\left(\int (1+|\xi|^2)|\widehat{u}(t,\xi)|^2 d\xi\right)^{\frac{1}{2}}\left(\int (1+|\xi|^2)e^{2\alpha|\xi|}|\widehat{u}(t,\xi)|^2 d\xi\right)^{\frac{1}{2}} \\ &\le e^{2c_3}\|u\|_{H^1}^{\frac{1}{2}}\|e^{a|D|}u(t)\|_{H^1}^{\frac{1}{2}} \\ &\le c\|u\|_{H^1}^{\frac{1}{2}}, \end{aligned}$$

where $c = e^{2c_3} c_0^{\frac{1}{2}}$.
Using (3.1), we get

$$\limsup_{t\to\infty} \|e^{a|D|^{1/\sigma}} u(t)\|_{H^1} = 0.$$

$\square$

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